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A THERMODYNAMICAL FORMULATION FOR DISPERSED MULTIPHASE TURBULENT FLOWS—I

BASIC THEORY

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Abstract—Turbulent flows of dispersed multiphase solid–fluid mixtures are considered. From the global equations of balance for each phase and via a special ensemble-averaging technique, the local conservation laws for the mean motions are developed. Particular attentions are given to the averaged form of the Clausius–Duhem inequality and the fluctuation energies of the fluid phase and the particulate constituents. The thermodynamics of the mixture in the turbulent state is studied. The concept of coldness of turbulence for each phase is introduced, the free energy function is discussed and several thermodynamical relationships are established. Based on the averaged entropy inequality, constitutive equations for the stresses, energy and heat fluxes of various species are developed. It is shown that the model contains the recently developed turbulence models for dilute two-phase flows and dense granular flows as special limiting cases.

Key Words: turbulent flow, multiphase mixture, thermodynamical model, granular flow

INTRODUCTION

Developing models capable of describing various multiphase flow regimes has attracted considerable attention due to their significant industrial applications. Extensive surveys of the literature on earlier and recent works were provided by Soo (1967), Ishii (1975), Hetsroni (1982), Drew (1983), Bedford & Drumheller (1983) and Ishii & Mishima (1984).

The existing theories of multiphase mixtures may be roughly classified into two categories. In one class the basic balance laws are postulated and continuum thermodynamics is used to arrive at appropriate constitutive laws. Typical examples are models developed by Passman *et al.* (1984) and Ahmadi (1980, 1982). In the other class, averaging techniques are employed to derive the fundamental balance laws. Several different averaging methods were used in the past. Drew (1983) and Ishii (1975) used the time-averaging technique, while Nigmatulin (1979), Hassanizadeh & Gray (1979) and Ahmadi (1987) employed the volume-averaging method. In many of the frequently used models (Soo 1967, 1981; Ishii 1975; Hetsroni 1982; Drew 1976, 1983; Ishii & Mishima 1984), the constitutive relationships were prescribed based on physical intuition and the restrictions imposed by the entropy inequality were not considered. Furthermore, the existing averaging models are limited to the first-moment equations. That is, only the mean of microscopic balance laws are considered and the higher order moments of these laws are totally ignored.

The importance of turbulence kinetic energy in turbulent flows of homogeneous fluids and its significance in developing turbulence models is now well-understood (Launder & Spalding 1972; Rodi 1982; Jones & Launder 1972; Launder *et al.* 1975; Lumley 1978, 1983; Ahmadi 1984). Modeling dispersed two-phase turbulent flows was considered more recently. Genchev & Karpuzov (1980), Taweel & Landau (1977) and Chen & Wood (1985) studied dilute gas–particle flows. In these models, the main assumption is that the particles are being simply transported by the carrier fluid flow. The effects of the particulate phase in modifying the fluid turbulence are, thus, totally ignored. Elghobashi & Abou-Arab (1983) developed a model which includes, in part, the interaction effects. However, their model neglects the particle kinetic and collisional effects and hence is still limited to relatively low concentration mixtures. Recently, Besnard & Harlow (1985) and Kashiwa (1987) proposed more elaborated models that offered certain improvements. Nevertheless, these models were not concerned with dense mixtures and particle collision effects.

The existing models for two-phase dispersed turbulent flows appear to be deficient in the following respects:

(i) Particle–particle collisional effects are generally neglected.
(ii) Only dilute mixtures are considered.
(iii) The effects of fluctuation kinetic energy of the particulate phase are neglected.
(iv) The interactions of the fluid and particulate phases are only partially considered.
(v) It is not clear if the closure assumptions used satisfy the required invariance and realizability conditions.
(vi) The requirements of the second law of thermodynamics are totally ignored.

In summary, an adequate model for describing two-phase turbulent flows of dense fluid–solid mixtures is not, as yet, available.

Recently, there have been interesting developments in modeling rapid flows of granular materials. The kinetic theory of gases was extended to model the motion of a dense collection of nearly elastic spherical particles (Ahmadi & Shahinpoor 1983a; Lun *et al.* 1984; Jenkins & Richman 1985; Ahmadi & Ma 1986). In the work of Ma & Ahmadi (1988), the effects of interstitial fluid were introduced in the kinetic model but the governing equations of motion of the fluid phase were not discussed.

Techniques similar to turbulence modeling for describing the motion of granular materials were introduced by Blinowski (1978), Ahmadi & Shahinpoor (1983b), Ahmadi (1985b) and Ma & Ahmadi (1985). In these latter works, a one-equation turbulence model for rapid flows of granular materials was developed which was shown to be consistent with the kinetic models for spherical nearly elastic particles. Ma & Ahmadi (1985) also showed that the model was capable of predicting the features of Couette and gravity rapid granular flows with reasonable accuracy.

Until very recently, the implications of the second law of thermodynamics in turbulence modeling were not studied. In the work of Ahmadi (1984, 1989), the averaged form of the entropy inequality for an incompressible fluid was derived and a thermodynamics for turbulence was developed. In this formulation, the turbulent fluctuation kinetic energy is treated as a second temperature, analogous to the molecular fluctuations which give rise to the thermodynamic temperature. Based on thermodynamical arguments, a two-equation model for turbulence was developed which resembles the well-known k–ϵ model. Comparing the predictions of the new model with experimental data, Busnaina *et al.* (1986) have shown that the model in its simplest form is superior to the standard k–ϵ model.

In this work, the global conservation laws for each phase are considered. The ensemble-averaging method is directly applied to the integral form of the balance laws. Favre's (1965) mass-weighted averaging for each phase is used and the global conservation laws for the mean motions of various species are derived. Using the divergence theorem, the local forms of the basic laws of motion for different constituents are developed. Particular attention is given to the formulation of the averaged Clausius–Duhem inequality. The equations governing the fluctuation kinetic energies of the particulate and fluid phases are derived. Based on the averaged entropy inequality, constitutive laws for the mean motions of different species are developed. It is shown that in the absence of the particulate phases, the model resembles the standard k–ϵ model of turbulence. When the fluid effects are neglected and only a single particulate phase is present, the model reduces to that obtained from the kinetic theories of granular materials. In the accompanying paper (Ma & Ahmadi 1990, this issue, pp. 341–351), the predictions of the model for simple shear flows of a dense mixture are compared with the experimental data.

GLOBAL BALANCE LAWS

Consider a dispersed mixture of n distinct particulate phases and a single fluid phase. The global balance laws for the αth phase in the multiphase mixture, in the absence of chemical reaction and interfacial mass transfer are given as:

conservation of mass,

$$\frac{\partial}{\partial t}\iiint_V \rho^\alpha \, \mathrm{d}V + \iint_A \rho^\alpha v_j^\alpha n_j \, \mathrm{d}A = 0; \tag{1}$$

balance of linear momentum,

$$\frac{\partial}{\partial t}\iiint_V \rho^\alpha v_i^\alpha \,\mathrm{d}V + \iint_A \rho^\alpha v_i^\alpha v_j^\alpha n_j \,\mathrm{d}A = \iiint_V \rho^\alpha f_i^\alpha \,\mathrm{d}V + \iint_A t_{ji}^\alpha n_j \,\mathrm{d}A + \iiint_V P_i^\alpha \,\mathrm{d}V; \quad [2]$$

balance of mechanical energy,

$$\frac{\partial}{\partial t}\iiint_V \rho^\alpha\left(\frac{v_i^\alpha v_i^\alpha}{2}\right)\mathrm{d}V + \iint_A \rho^\alpha\left(\frac{v_i^\alpha v_i^\alpha}{2}\right)v_j^\alpha n_j \,\mathrm{d}A$$
$$= \iiint_V \rho^\alpha v_i^\alpha f_i^\alpha \,\mathrm{d}V + \iiint_V v_i^\alpha t_{ji,j}^\alpha \,\mathrm{d}V + \iiint_V v_i^\alpha P_i^\alpha \,\mathrm{d}V; \quad [3]$$

conservation of energy,

$$\frac{\partial}{\partial t}\iiint_V \rho^\alpha\left(\frac{v_i^\alpha v_i^\alpha}{2} + e^\alpha\right)\mathrm{d}V + \iint_A \rho^\alpha\left(\frac{v_i^\alpha v_i^\alpha}{2} + e^\alpha\right)v_j^\alpha n_j \,\mathrm{d}A$$
$$= \iiint_V \rho^\alpha v_i^\alpha f_i^\alpha \,\mathrm{d}V + \iint_A v_i^\alpha t_{ji}^\alpha n_j \,\mathrm{d}A + \iint_A q_i^\alpha n_i \,\mathrm{d}A + \iiint_V (r^\alpha + e^{\alpha+})\,\mathrm{d}V; \quad [4]$$

and

Clausius–Duhem inequality,

$$\sum_{\alpha=1}^{n+1}\left[\frac{\partial}{\partial t}\iiint_V \rho^\alpha \eta^\alpha \,\mathrm{d}V + \iint_A \rho^\alpha \eta^\alpha v_j^\alpha n_j \,\mathrm{d}A - \iint_A q_i^\alpha \vartheta^\alpha n_i \,\mathrm{d}A - \iiint_V (r^\alpha \vartheta^\alpha + \eta^{\alpha+})\,\mathrm{d}V\right] \geqslant 0. \quad [5]$$

In these equations, V is a fixed volume of space with surface A, $\mathbf{v}$ is the instantaneous velocity vector, ρ is the density, $\mathbf{n}$ is the unit normal vector, $\mathbf{f}$ is the body force per unit mass, t_{ji} is the stress tensor, P_i is the interaction momentum supply per unit volume, e is the internal energy per unit mass, q_i is the heat flux vector pointing outward of an enclosed volume, r is the internal heat source per unit volume, e^+ is the interaction energy supply, η is the entropy per unit mass, η^+ is the interaction entropy supply and ϑ is the coldness, defined as

$$\vartheta = \frac{1}{\theta} \quad [6]$$

where θ is the temperature. The superscript $\alpha\,(1 \leqslant \alpha \leqslant n)$ represents the αth particulate phase and $\alpha = n + 1$ (superscript f is used later) denotes the fluid phase. Note that all these field quantities have highly irregular and discontinuous distributions due to the corpuscular nature of the particulate phases. Throughout this work the regular Cartesian tensor notation with Latin subscripts is used. Thus, indices after a comma denote partial derivatives and $\mathrm{d}/\mathrm{d}t$ stands for the total time derivative.

In a state of turbulent motion, the field quantities for all phases behave randomly and fluctuate vigorously. Each random function could be expressed as a sum of a mean and a fluctuating part, i.e.

$$\begin{aligned}
\rho^\alpha &= \bar{\rho}^\alpha + \rho^{\alpha'}, & \overline{\rho^{\alpha'}} &= 0,\\
v_i^\alpha &= \bar{v}_i^\alpha + v_i^{\alpha'}, & \overline{v_i^{\alpha'}} &= 0,\\
P_i^\alpha &= \bar{P}_i^\alpha + P_i^{\alpha'}, & \overline{P_i^{\alpha'}} &= 0,\\
\theta^\alpha &= \bar{\theta}^\alpha + \theta^{\alpha'}, & \overline{\theta^{\alpha'}} &= 0,\\
\vartheta^\alpha &= \bar{\vartheta}^\alpha + \vartheta^{\alpha'}, & \overline{\vartheta^{\alpha'}} &= 0,\\
t_{ij}^\alpha &= \bar{t}_{ij}^\alpha + t_{ij}^{\alpha'}, & \overline{t_{ij}^{\alpha'}} &= 0,\\
q_i^\alpha &= \bar{q}_i^\alpha + q_i^{\alpha'}, & \overline{q_i^{\alpha'}} &= 0,\\
e^{\alpha+} &= \bar{e}^{\alpha+} + e^{\alpha+'}, & \overline{e^{\alpha+'}} &= 0,\\
\eta^{\alpha+} &= \bar{\eta}^{\alpha+} + \eta^{\alpha+'}, & \overline{\eta^{\alpha+'}} &= 0.
\end{aligned} \quad [7]$$

Here, a bar over a character stands for the expected value (ensemble average) and a prime denotes the fluctuating part. The body force and heat source are assumed to be nonfluctuating. As pointed out by Favre (1965), it is possible to obtain convenient forms of the equations governing the mean motion by introducing a mass-weighted ensemble averaging for certain field quantities (Ahmadi & Shahinpoor 1983b; Ahmadi 1989). Accordingly, the following decompositions are introduced:

$$
\begin{aligned}
v_i^\alpha &= \tilde{v}_i^\alpha + v_i^{\alpha''}, \qquad \tilde{v}_i^\alpha = \frac{\overline{\rho^\alpha v_i^\alpha}}{\bar{\rho}^\alpha}, \quad \overline{v_i^{\alpha''}} = \frac{\overline{\rho^\alpha v_i^{\alpha'}}}{\bar{\rho}^\alpha}, \\
e^\alpha &= \tilde{e}^\alpha + e^{\alpha''}, \qquad \tilde{e}^\alpha = \frac{\overline{\rho^\alpha e^\alpha}}{\bar{\rho}^\alpha}, \\
\eta^\alpha &= \tilde{\eta}^\alpha + \eta^{\alpha''}, \qquad \tilde{\eta}^\alpha = \hat{\eta}^\alpha + \eta^{\alpha\mathrm{T}} = \frac{\overline{\rho^\alpha \eta^\alpha}}{\bar{\rho}^\alpha}.
\end{aligned}
\qquad [8]
$$

Here, a tilde over a character represents a mass-weighted ensemble-averaged quantity and a double prime stands for the fluctuating part relative to the mass-weighted averaged magnitude. Equation [8] also shows that $\overline{v_i^{\alpha''}}$ is proportional to the fluctuation velocity–density correlation. Note that the ensemble average of a double prime quantity is not zero, while

$$
\overline{\rho^\alpha v_i^{\alpha''}} = \overline{\rho^\alpha e^{\alpha''}} = \overline{\rho^\alpha \eta^{\alpha''}} = 0. \qquad [9]
$$

The decomposition of the mass-weighted averaged entropy needs further explanation. In line with Ahmadi (1984, 1989), it is assumed that the mean entropy $\tilde{\eta}^\alpha$ is composed of two parts, $\hat{\eta}^\alpha$ and $\eta^{\alpha\mathrm{T}}$. These mean entropies correspond to the molecular and turbulent agitations, respectively. While $\hat{\eta}^\alpha$ is a function of temperature, $\eta^{\alpha\mathrm{T}}$ is expected to be a function of the state of fluctuation (turbulence) of the αth phase. Thus, for an isothermal turbulent flow, $\hat{\eta}^\alpha$ is a constant while $\eta^{\alpha\mathrm{T}}$ is a variable.

Taking the ensemble average of [1]–[5], and using the decompositions given by [7] and [8], the integral form of the balance laws follows. These are:

conservation of mass,

$$
\frac{\partial}{\partial t}\iiint_V \bar{\rho}^\alpha \,\mathrm{d}V + \iint_A \bar{\rho}^\alpha \tilde{v}_j^\alpha n_j \,\mathrm{d}A = 0; \qquad [10]
$$

balance of linear momentum,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\iiint_V \bar{\rho}^\alpha \tilde{v}_i^\alpha \,\mathrm{d}V + \iint_A \bar{\rho}^\alpha \tilde{v}_i^\alpha \tilde{v}_j^\alpha n_j \,\mathrm{d}A + \iint_A \overline{\rho^\alpha v_i^{\alpha''} v_j^{\alpha''}} n_j \,\mathrm{d}A \\
&\qquad = \iiint_V \bar{\rho}^\alpha f_i^\alpha \,\mathrm{d}V + \iint_A \bar{t}_{ji}^\alpha n_j \,\mathrm{d}A + \iiint_V \bar{P}_i^\alpha \,\mathrm{d}V; \qquad [11]
\end{aligned}
$$

balance of mechanical energy,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\iiint_V \bar{\rho}^\alpha\left(\frac{\tilde{v}_i^\alpha \tilde{v}_i^\alpha}{2} + k^\alpha\right)\mathrm{d}V + \iint_A \bar{\rho}^\alpha\left(\frac{\tilde{v}_i^\alpha \tilde{v}_i^\alpha}{2} + k^\alpha\right)\tilde{v}_j^\alpha n_j \,\mathrm{d}A \\
&\quad + \iint_A \overline{\rho^\alpha \frac{v_i^{\alpha''} v_i^{\alpha''}}{2} v_j^{\alpha''}} n_j \,\mathrm{d}A + \iint_A \overline{\rho^\alpha v_i^{\alpha''} v_j^{\alpha''}}\, \tilde{v}_i^\alpha n_j \,\mathrm{d}A \\
&= \iiint_V \bar{\rho}^\alpha \tilde{v}_i^\alpha f_i^\alpha \,\mathrm{d}V + \iiint_V \tilde{v}_i^\alpha \bar{t}_{ji,j}^\alpha \,\mathrm{d}V + \iiint_V \overline{v_i^{\alpha''}}\, \bar{t}_{ji,j}^\alpha \,\mathrm{d}V + \iiint_V \overline{v_i^{\alpha''} t_{ji,j}^{\alpha'}} \,\mathrm{d}V + \iiint_V \overline{v_i^\alpha P_i^\alpha} \,\mathrm{d}V; \qquad [12]
\end{aligned}
$$

conservation of energy,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\iiint_V \bar{\rho}^\alpha\left(\frac{\tilde{v}_i^\alpha \tilde{v}_i^\alpha}{2} + k^\alpha + \tilde{e}^\alpha\right)\mathrm{d}V + \iint_A \bar{\rho}^\alpha\left(\frac{\tilde{v}_i^\alpha \tilde{v}_i^\alpha}{2} + k^\alpha + \tilde{e}^\alpha\right)\tilde{v}_j^\alpha n_j \,\mathrm{d}A + \iint_A \overline{\rho^\alpha \frac{\tilde{v}_i^{\alpha''} \tilde{v}_i^{\alpha''}}{2} v_j^{\alpha''}} n_j \mathrm{d}A \\
&\quad + \iint_A \overline{\rho^\alpha v_i^{\alpha''} v_j^{\alpha''}}\, \tilde{v}_i^\alpha n_j \,\mathrm{d}A + \iint_A \overline{\rho^\alpha e^{\alpha''} v_j^{\alpha''}} n_j \,\mathrm{d}A \\
&= \iiint_V \bar{\rho}^\alpha \tilde{v}_i^\alpha f_i^\alpha \,\mathrm{d}V + \iint_A \tilde{v}_i^\alpha \bar{t}_{ji}^\alpha n_j \,\mathrm{d}A + \iint_A \overline{v_i^{\alpha''}}\, \bar{t}_{ji}^{\alpha'} n_j \,\mathrm{d}A + \iint_A v_i^{\alpha''} \bar{t}_{ji}^{\alpha'} n_j \,\mathrm{d}A \\
&\quad + \iint_A \bar{q}_i^\alpha n_i \,\mathrm{d}A + \iiint_V (r^\alpha + \bar{e}^{\alpha+}) \,\mathrm{d}V. \qquad [13]
\end{aligned}
$$

and

Clausius–Duhem inequality,

$$\sum_{\alpha=1}^{n+1}\left[\frac{\partial}{\partial t}\iiint_V \bar{\rho}^\alpha \tilde{\eta}^\alpha \,\mathrm{d}V + \iint_A (\bar{\rho}^\alpha \tilde{\eta}^\alpha \tilde{v}_j^\alpha + \overline{\rho^\alpha \eta^{\alpha''} v_j^{\alpha''}}) n_j \,\mathrm{d}A - \iint_A (\bar{q}_i^\alpha \bar{\vartheta}^\alpha + \overline{q_i^{\alpha'} \vartheta^{\alpha'}}) n_i \,\mathrm{d}A - \iiint_V (r^\alpha \bar{\vartheta}^\alpha + \bar{\eta}^{\alpha+}) \,\mathrm{d}V\right] \geqslant 0. \quad [14]$$

In these equations, k^α is the fluctuation kinetic energy of the αth phase defined as

$$\bar{\rho}^\alpha k^\alpha = \overline{\rho^\alpha \frac{v_i^{\alpha''} v_i^{\alpha''}}{2}}. \quad [15]$$

Equations [10]–[14] are the global conservation laws for the mean motion of the multiphase mixture.

DIFFERENTIAL BALANCE LAWS

Even though the instantaneous field variables in a dispersed multiphase turbulent flow are highly irregular, discontinuous and nondifferentiable functions, their averages are continuous, smoothly varying, differentiable functions of space and time. Therefore, the integrands in [10]–[14] are well-behaved functions and the integral theorems could be used. Applying the divergence theorem to the surface integrals in [10]–[14] and rearranging terms, the differential balance laws for the mean motion of the multiphase mixture follow.

The differential forms of the equations of the conservation of mass for the αth particulate phase and for the fluid phase, as obtained from [10], are given as

$$\frac{\partial \bar{\rho}^\alpha}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}^\alpha \tilde{v}_j^\alpha) = 0 \quad [16]$$

and

$$\frac{\partial \bar{\rho}^\mathrm{f}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}^\mathrm{f} \tilde{v}_j^\mathrm{f}) = 0. \quad [17]$$

When the αth particulate phase is incompressible, it follows that

$$\bar{\rho}^\alpha = \rho^\alpha v^\alpha, \quad [18]$$

where v^α is the mean volume fraction of the αth phase. Since ρ^α is a constant, [16] may now be restated as

$$\frac{\partial v^\alpha}{\partial t} + \frac{\partial}{\partial x_j}(v^\alpha \tilde{v}_j^\alpha) = 0. \quad [19]$$

Similarly, when the fluid phase is incompressible with constant density ρ^f, we find

$$\bar{\rho}^\mathrm{f} = \rho^\mathrm{f} v^\mathrm{f} \quad [20]$$

and [17] may be rewritten as

$$\frac{\partial v^\mathrm{f}}{\partial t} + \frac{\partial}{\partial x_j}(v^\mathrm{f} \tilde{v}_j^\mathrm{f}) = 0. \quad [21]$$

For fully-saturated mixtures, the following constraint is imposed:

$$v^\mathrm{f} + \sum_{\alpha=1}^{n} v^\alpha = 1. \quad [22]$$

Equation [11] leads to the local forms of the balance of linear momentum for the αth particulate phase and for the fluid phase. These are

$$\bar{\rho}^\alpha \frac{\mathrm{d}\tilde{v}_i^\alpha}{\mathrm{d}t} = \bar{\rho}^\alpha f_i^\alpha + \frac{\partial \bar{\tau}_{ji}^\alpha}{\partial x_j} + \frac{\partial \hat{t}_{ji}^\alpha}{\partial x_j} + \bar{P}_i^\alpha \quad [23]$$

and

$$\bar{\rho}^{\mathrm{f}} \frac{\mathrm{d}\tilde{v}_i^{\mathrm{f}}}{\mathrm{d}t} = \bar{\rho}^{\mathrm{f}} f_i^{\mathrm{f}} + \frac{\partial \bar{t}_{ji}^{\mathrm{f}}}{\partial x_j} + \frac{\partial \hat{t}_{ji}^{\mathrm{fT}}}{\partial x_j} + \bar{P}_i^{\mathrm{f}}. \qquad [24]$$

Note that for the particulate phases, the fluctuation (kinetic) stress tensor $t_{ji}^{\alpha\mathrm{T}}$ and the collisional stress tensor $t_{ji}^{\alpha\mathrm{c}}$ are combined, i.e.

$$\bar{\tau}_{ji}^{\alpha} = \bar{t}_{ji}^{\alpha\mathrm{c}} + t_{ji}^{\alpha\mathrm{T}}, \quad t_{ji}^{\alpha\mathrm{T}} = -\overline{\rho^{\alpha} v_j^{\alpha''} v_i^{\alpha''}}; \qquad [25]$$

and

$$\hat{t}_{ji}^{\alpha} = \bar{t}_{ji}^{\alpha} - \bar{t}_{ji}^{\alpha\mathrm{c}}, \qquad [26]$$

is the average stress tensor in the absence of collisional effects. The fluid turbulent stress tensor t_{ji}^{fT} is defined as

$$t_{ji}^{\mathrm{fT}} = -\overline{\rho^{\mathrm{f}} v_j^{\mathrm{f}''} v_i^{\mathrm{f}''}}. \qquad [27]$$

Based on [16] and [17], the equation of the conservation mass for the entire mixture becomes

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_j}(\bar{\rho}\tilde{v}_j) = 0, \qquad [28]$$

where the density of mixture $\bar{\rho}$ and the velocity of mixture $\tilde{v}_j$ are defined as

$$\bar{\rho} = \bar{\rho}^{\mathrm{f}} + \sum_{\alpha=1}^{n} \bar{\rho}^{\alpha}, \quad \bar{\rho}\tilde{v}_j = \bar{\rho}^{\mathrm{f}}\tilde{v}_j^{\mathrm{f}} + \sum_{\alpha=1}^{n} \bar{\rho}^{\alpha}\tilde{v}_j^{\alpha}. \qquad [29]$$

Similarly, the balance of linear momentum for the entire mixture follows by adding [23] and [24] for all the species, i.e.

$$\bar{\rho}\frac{\mathrm{d}\tilde{v}_i}{\mathrm{d}t} = \bar{\rho} f_i + \frac{\partial \tau_{ji}}{\partial x_j}, \qquad [30]$$

where the net stress is defined as

$$\tau_{ji} = \bar{t}_{ji}^{\mathrm{f}} + t_{ji}^{\mathrm{fT}} - \bar{\rho}^{\mathrm{f}}\tilde{v}_i^{\mathrm{f}}\tilde{v}_j^{\mathrm{f}} + \sum_{\alpha=1}^{n}(\bar{\tau}_{ji}^{\alpha} + \hat{t}_{ji}^{\alpha} - \bar{\rho}^{\alpha}\tilde{v}_i^{\alpha}\tilde{v}_j^{\alpha}) + \bar{\rho}\tilde{v}_i\tilde{v}_j. \qquad [31]$$

In the derivation of [30] we have used the condition that the net interaction momentum supply must be zero, i.e.

$$\bar{P}_i^{n+1} = \bar{P}_i^{\mathrm{f}} = -\sum_{\alpha=1}^{n} \bar{P}_i^{\alpha}. \qquad [32]$$

The balance of mechanical energy, as given by [12], may be restated in differential form as

$$\bar{\rho}^{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\tilde{v}_i^{\alpha}\tilde{v}_i^{\alpha}}{2}\right) + \bar{\rho}^{\alpha}\frac{\mathrm{d}k^{\alpha}}{\mathrm{d}t} = -\frac{\partial}{\partial x_j}\left(\overline{\frac{\rho^{\alpha} v_i^{\alpha''} v_i^{\alpha''}}{2} v_j^{\alpha''}}\right) - \frac{\partial}{\partial x_j}(\overline{\rho^{\alpha} v_i^{\alpha''} v_j^{\alpha''}}\tilde{v}_i^{\alpha})$$
$$+ \bar{\rho}^{\alpha}\tilde{v}_i^{\alpha} f_i^{\alpha} + \tilde{v}_i^{\alpha}\bar{t}_{ji,j}^{\alpha} + \overline{v_i^{\alpha''}}\bar{t}_{ji,j}^{\alpha} + \overline{v_i^{\alpha''} t_{ji,j}^{\alpha'}} + \overline{v_i^{\alpha} P_i^{\alpha}}. \qquad [33]$$

Multiplying [23] by $\tilde{v}_i^{\alpha}$ and subtracting the result from [33], the equation governing the evolution of the fluctuation energy of the αth particulate phase follows:

$$\bar{\rho}^{\alpha}\frac{\mathrm{d}k^{\alpha}}{\mathrm{d}t} = -\frac{\partial}{\partial x_j}\left(\overline{\frac{\rho^{\alpha} v_i^{\alpha''} v_i^{\alpha''}}{2} v_j^{\alpha''}} - \overline{v_i^{\alpha''} t_{ji}^{\alpha'}}\right) - \overline{\rho^{\alpha} v_i^{\alpha''} v_j^{\alpha''}}\tilde{v}_{i,j}^{\alpha} - \overline{t_{ji}^{\alpha'} v_{i,j}^{\alpha''}} + \overline{v_i^{\alpha''}}\bar{t}_{ji,j}^{\alpha} + \overline{v_i^{\alpha} P_i^{\alpha}} - \bar{P}_i^{\alpha}\tilde{v}_i^{\alpha}. \qquad [34]$$

Equation [34] may be rewritten in a compact form as

$$\bar{\rho}^{\alpha}\frac{\mathrm{d}k^{\alpha}}{\mathrm{d}t} = \bar{\tau}_{ji}^{\alpha}\tilde{v}_{i,j}^{\alpha} - \overline{v_i^{\alpha''}}\bar{p}_{,i}^{\alpha} + K_{j,j}^{\alpha} - \bar{\rho}^{\alpha}\epsilon^{\alpha} + \bar{\rho}^{\alpha}s^{\alpha}, \qquad [35]$$

where

$$\bar{\rho}^{\alpha}\epsilon^{\alpha} = \bar{\rho}^{\alpha}(\epsilon^{\alpha\mathrm{c}} + \epsilon^{\alpha\mathrm{v}}), \quad \bar{\rho}^{\alpha}\epsilon^{\alpha\mathrm{c}} = \overline{t_{ij}^{\alpha\mathrm{c}} v_{i,j}^{\alpha}}, \quad \bar{\rho}^{\alpha}\epsilon^{\alpha\mathrm{v}} = \overline{t_{ij}^{\alpha\mathrm{v}} v_{i,j}^{\alpha''}}. \qquad [36]$$

Here, ϵ^α is the dissipation rate for the αth phase per unit mass (Harlow & Nakayama 1967; Rodi 1982), $\epsilon^{\alpha c}$ and $\epsilon^{\alpha v}$ are the particulate collisional and viscous dissipation rates (Ahmadi 1985b) and

$$t_{ij}^{\alpha v} = t_{ij}^{\alpha} + p^\alpha \delta_{ij} \qquad [37]$$

is the viscous (dissipative) part of t_{ij}^α. For nearly elastic particles $\epsilon^{\alpha v}$ is negligible, while for relatively dilute particulate concentration $\epsilon^{\alpha c}$ may be neglected. In [35],

$$K_j^\alpha = -\overline{\rho^\alpha \frac{v_i^{\alpha''} v_i^{\alpha''}}{2} v_j^{\alpha''}} + \overline{v_i^{\alpha''} t_{ji}^{\alpha v}} + \overline{v_i^{\alpha''} t_{ji}^{\alpha c}} - \overline{v_j^{\alpha''} p^{\alpha'}} \qquad [38]$$

is the particulate fluctuation energy flux vector and

$$\bar{\rho}^\alpha s^\alpha = \overline{v_i^\alpha P_i^\alpha} - \bar{P}_i^\alpha \tilde{v}_i^\alpha = \overline{v_i^{\alpha''} P_i^\alpha} \qquad [39]$$

is the particulate interaction fluctuation energy supply term. Note that in [35], the particulate fluctuation pressure–velocity gradient correlation term $\overline{v_{i,i}^{\alpha''} p^{\alpha'}}$ was neglected. Equation [35] clearly shows that the particulate fluctuation energy is being produced by the action of the total fluctuation (kinetic + collisional) stresses in a mean shear field and is being transported by convection and diffusion and is being dissipated. Equation [35] also shows that fluctuation energy could be supplied or extracted through the interaction source term. There is also a secondary source term related to the product of the density–velocity correlation and mean pressure gradient field.

For the fluid phase, [34] may be restated as

$$\bar{\rho}^f \frac{dk^f}{dt} = t_{ji}^{fT} \tilde{v}_{i,j}^f - \overline{v_i^{f''}} \bar{p}_{,i}^f + K_{j,j}^f - \bar{\rho}^f \epsilon^f + \bar{\rho}^f s^f, \qquad [40]$$

where the fluid turbulent kinetic energy k^f and fluid dissipation rate ϵ^f are defined as

$$\bar{\rho}^f k^f = \overline{\rho^f \frac{v_i^{f''} v_i^{f''}}{2}} \qquad [41]$$

and

$$\bar{\rho}^f \epsilon^f = \overline{t_{ji}^{fv} v_{i,j}^{f''}}, \qquad [42]$$

respectively. Here,

$$K_j^f = -\overline{\rho^f \frac{v_i^{f''} v_i^{f''}}{2} v_j^{f''}} + \overline{v_i^{f''} t_{ji}^{fv}} - \overline{v_j^{f''} p^{f'}}, \qquad [43]$$

is the fluid fluctuation energy flux vector and

$$\bar{\rho}^f s^f = \overline{v_i^f P_i^f} - \bar{P}_i^f \tilde{v}_i^f = \overline{v_i^{f''} P_i^f}, \qquad [44]$$

is the fluid fluctuation energy supply term. The instantaneous stress tensor in the fluid phase is expressed as

$$t_{ji}^f = -p^f \delta_{ij} + t_{ji}^{fv}, \qquad [45]$$

where t_{ji}^{fv} is the viscous part of the fluid stress tensor and p^f is the fluid hydrodynamic pressure. Like the particulate case, the terms corresponding to production, diffusion, dissipation and interaction supply of fluid fluctuation energy can be identified in [40]. Here, the fluctuation velocity–pressure gradient correlation term $\overline{v_{i,i}^{f''} p^{f'}}$ was also neglected.

Subtracting the mechanical energy equation given by [12] from [13], we find

$$\frac{\partial}{\partial t}\iiint_V \bar{\rho}^\alpha \tilde{e}^\alpha \, dV + \iint_A \bar{\rho}^\alpha \tilde{e}^\alpha \tilde{v}_j^\alpha n_j \, dA + \iint_A \overline{\rho^\alpha e^{\alpha''} v_j^{\alpha''}} n_j \, dA$$
$$= \iiint_V \overline{t_{ji}^\alpha v_{j,i}^\alpha} \, dV + \iint_A \bar{q}_i^\alpha n_i \, dA - \iiint_V \overline{v_i^\alpha P_i^\alpha} \, dV + \iiint_V (r^\alpha + \bar{e}^{\alpha +}) \, dV. \qquad [46]$$

Applying the divergence theorem to the surface integrals in [46], we find

$$\bar{\rho}^\alpha \frac{d\tilde{e}^\alpha}{dt} = \bar{q}_{j,j}^\alpha + q_{j,j}^{\alpha T} + \bar{\rho}^\alpha \epsilon^\alpha + \hat{t}_{ji}^\alpha \tilde{v}_{j,i}^\alpha - \overline{v_i^\alpha P_i^\alpha} - \bar{p}^\alpha \overline{v_{i,i}^{\alpha''}} + r^\alpha + \bar{e}^{\alpha +}, \qquad [47]$$

where

$$q_j^{\alpha T} = \overline{\rho^\alpha e^{\alpha''} v_j^{\alpha''}} \qquad [48]$$

is the αth particulate turbulent heat flux vector. Equation [47] is the statement of the local form of the conservation of energy for the αth particulate phase.

Similarly, the equation for the conservation of energy for the fluid phase in differential form becomes

$$\bar{\rho}^f \frac{d\tilde{e}^f}{dt} = \bar{q}_{j,j}^f + q_{j,j}^{fT} + \bar{\rho}^f \epsilon^f + \bar{t}_{ji}^f \tilde{v}_{j,i} - \overline{v_i^f P_i^f} - \bar{p}^f \overline{v_{i,i}^{f''}} + r^f + \bar{e}^{f+}, \qquad [49]$$

where

$$q_j^{fT} = \overline{\rho^f e^{f''} v_j^{f''}} \qquad [50]$$

is the fluid turbulent heat flux vector. Note that in [47] and [49], the fluctuation pressure–velocity gradient correlation terms $\overline{p^{\alpha'} v_{i,i}^{\alpha''}}$ and $\overline{p^{f'} v_{i,i}^{f''}}$ were neglected.

Using the divergence theorem in the entropy inequality equation given by [14], the differential form of the Clausius–Duhem inequality follows. The results may be restated as

$$\sum_{\alpha=1}^{n+1} \left[\bar{\rho}^\alpha \dot{\hat{\eta}}^\alpha - (\bar{q}_i^\alpha \mathfrak{J}^\alpha)_{,i} - R_{i,i}^{\alpha T} - r^\alpha \mathfrak{J}^\alpha + \bar{\rho}^\alpha \dot{\eta}^{\alpha T} - S_{i,i}^{\alpha T} - \bar{\eta}^{\alpha +} \right] \geqslant 0, \qquad [51]$$

where the turbulent entropy flux vector $S_i^{\alpha T}$ and the heat flux–coldness correlation vector $R_i^{\alpha T}$ are defined as

$$S_i^{\alpha T} = -\overline{\rho^\alpha v_i^{\alpha''} \eta^{\alpha''}} \qquad [52]$$

and

$$R_i^{\alpha T} = \overline{q_i^{\alpha'} \vartheta^{\alpha'}}. \qquad [53]$$

We now introduce the Helmholtz free energy functions for each phase for the mean thermal and turbulent fluctuations as

$$\hat{\psi}^\alpha = \tilde{e}^\alpha - \frac{\hat{\eta}^\alpha}{\mathfrak{J}^\alpha}, \quad \psi^{\alpha T} = k^\alpha - \frac{\eta^{\alpha T}}{\vartheta^{\alpha T}},$$
$$\hat{\psi}^f = \tilde{e}^f - \frac{\hat{\eta}^f}{\mathfrak{J}^f}, \quad \psi^{fT} = k^f - \frac{\eta^{fT}}{\vartheta^{fT}}. \qquad [54]$$

Here, $\vartheta^{\alpha T}$ and ϑ^{fT} are the particulate fluctuation and the fluid turbulence coldness defined analogous to the thermal coldness. Their precise expressions will be introduced in the next section. Here, it suffices to point out that, based on the disparity of the scales of molecular fluctuations and turbulence agitations, independent coldness and free energy functions, given by [54], are considered. Furthermore, the kinetic theories of single-phase monoatomic gases and nearly elastic spherical granular particles may be used to justify the forms given by [54]. It should be emphasized that the underlying assumption for treating phasic turbulence kinetic energies as thermodynamical variables is their persistence on the macroscopic time scales of interest.

Using [35], [40], [47], [49] and [54] in [51], the result may be restated as

$$\sum_{\alpha=1}^{n} \mathfrak{J}^\alpha \Bigg[-\bar{\rho}^\alpha \left(\dot{\hat{\psi}}^\alpha - \hat{\eta}^\alpha \frac{\dot{\mathfrak{J}}^\alpha}{(\mathfrak{J}^\alpha)^2} \right) + \bar{q}_{j,j}^\alpha + q_{j,j}^{\alpha T} + \hat{t}_{ji}^\alpha \tilde{v}_{j,i}^\alpha + \bar{\rho}^\alpha \epsilon^\alpha$$
$$- \frac{1}{\mathfrak{J}^\alpha} (\bar{q}_i^\alpha \mathfrak{J}^\alpha)_{,i} - \tilde{v}_i^\alpha \bar{P}_i^\alpha - \bar{\rho}^\alpha s^\alpha - \bar{p}^\alpha \overline{v_{i,i}^{\alpha''}} - \frac{1}{\mathfrak{J}^\alpha} R_{i,i}^{\alpha T} + \bar{e}^{\alpha +} - \frac{1}{\mathfrak{J}^\alpha} \bar{\eta}^{\alpha +} \Bigg]$$
$$+ \mathfrak{J}^f \Bigg[-\bar{\rho}^f \left(\dot{\hat{\psi}}^f - \hat{\eta}^f \frac{\dot{\mathfrak{J}}^f}{(\mathfrak{J}^f)^2} \right) + \bar{q}_{j,j}^f + q_{j,j}^{fT} + \bar{t}_{ji}^f \tilde{v}_{j,i}^f + \bar{\rho}^f \epsilon^f$$
$$- \frac{1}{\mathfrak{J}^f} (\bar{q}_i^f \mathfrak{J}^f)_{,i} - \tilde{v}_i^f \bar{P}_i^f - \bar{\rho}^f s^f - \bar{p}^f \overline{v_{i,i}^{f''}} - \frac{1}{\mathfrak{J}^f} R_{i,i}^{fT} + \bar{e}^{f+} - \frac{1}{\mathfrak{J}^f} \bar{\eta}^{f+} \Bigg]$$

$$+\sum_{\alpha=1}^{n} \vartheta^{\alpha T}\left[-\bar{\rho}^{\alpha}\left(\dot{\psi}^{\alpha T}-\eta^{\alpha T}\frac{\dot{\vartheta}^{\alpha T}}{(\vartheta^{\alpha T})^2}\right)+\bar{\tau}^{\alpha}_{ji}\tilde{v}^{\alpha}_{i,j}-\bar{\rho}^{\alpha}\epsilon^{\alpha}-\overline{v_i^{\alpha''}}\bar{p}^{\alpha}_{,i}+K^{\alpha}_{j,j}+\bar{\rho}^{\alpha}s^{\alpha}-\frac{1}{\vartheta^{\alpha T}}S^{\alpha T}_{i,i}\right]$$
$$+\vartheta^{fT}\left[-\bar{\rho}^{f}\left(\dot{\psi}^{fT}-\eta^{fT}\frac{\dot{\vartheta}^{fT}}{(\vartheta^{fT})^2}\right)+t^{fT}_{ji}\tilde{v}^{f}_{i,j}-\bar{\rho}^{f}\epsilon^{f}-\overline{v_i^{f''}}\bar{p}^{f}_{,i}+K^{f}_{j,j}+\bar{\rho}^{f}s^{f}-\frac{1}{\vartheta^{fT}}S^{fT}_{i,i}\right]\geqslant 0, \quad [55]$$

where [39] and [44] are also used. Equation [55] is the averaged form of the Clausius–Duhem inequality for turbulent multiphase flows.

CONSTITUTIVE EQUATIONS

To develop a model for the mean turbulent motion of a single-phase fluid, the mean field is essentially treated as a non-Newtonian and multitemperature material (Lumley 1970; Ahmadi 1985a). In the present formulation, we are concerned with multiphase turbulent flows. The mean field equations governing the mixture motion are given by [16], [17], [23], [24], [35], [40], [47], [49] and [55]. In this section, based on the averaged entropy inequality given by [55], a set of constitutive equations for the mean turbulent multiphase flow will be developed. Following Ahmadi (1989), in analogy with classical thermodynamics and kinetic theories of monoatomic gases and granular materials, we assume

$$R^{\alpha T}_i = q^{\alpha T}_i \mathfrak{J}^{\alpha}, \quad S^{\alpha T}_i = K^{\alpha}_i \vartheta^{\alpha T},$$
$$R^{fT}_i = q^{fT}_i \mathfrak{J}^{f}, \quad S^{fT}_i = K^{f}_i \vartheta^{fT} - E^{f}_i. \quad [56]$$

That is, an entropy flux is assumed to be equal to the product of the energy flux vector and the corresponding coldness. For the fluid phase, a vector $\mathbf{E}^f$ is introduced in [56] to account for the possible differences. It may be shown that $\mathbf{E}^f$ is related to the diffusion of the dissipation rate (Ahmadi 1989). Using [56], [55] may be restated as

$$\sum_{\alpha=1}^{n} \mathfrak{J}^{\alpha}\left[-\bar{\rho}^{\alpha}\left(\hat{\dot{\psi}}^{\alpha}-\hat{\eta}^{\alpha}\frac{\dot{\mathfrak{J}}^{\alpha}}{(\mathfrak{J}^{\alpha})^2}\right)-\frac{1}{\mathfrak{J}^{\alpha}}Q^{\alpha}_i\mathfrak{J}^{\alpha}_{,i}+\hat{t}^{\alpha}_{ji}\tilde{v}^{\alpha}_{j,i}+\bar{\rho}^{\alpha}\epsilon^{\alpha}-\tilde{v}^{\alpha}_i\bar{P}^{\alpha}_i-\bar{\rho}^{\alpha}s^{\alpha}-\bar{p}^{\alpha}\overline{v^{\alpha''}_{i,i}}+\bar{e}^{\alpha+}-\frac{1}{\mathfrak{J}^{\alpha}}\bar{\eta}^{\alpha+}\right]$$
$$+\mathfrak{J}^{f}\left[-\bar{\rho}^{f}\left(\hat{\dot{\psi}}^{f}-\hat{\eta}^{f}\frac{\dot{\mathfrak{J}}^{f}}{(\mathfrak{J}^{f})^2}\right)-\frac{1}{\mathfrak{J}^{f}}Q^{f}_i\mathfrak{J}^{f}_{,i}+\bar{t}^{f}_{ji}\tilde{v}^{f}_{j,i}+\bar{\rho}^{f}\epsilon^{f}-\tilde{v}^{f}_i\bar{P}^{f}_i-\bar{\rho}^{f}s^{f}-\bar{p}^{f}\overline{v^{f''}_{i,i}}+\bar{e}^{f+}-\frac{1}{\mathfrak{J}^{f}}\bar{\eta}^{f+}\right]$$
$$+\sum_{n=1}^{\alpha} \vartheta^{\alpha T}\left[-\bar{\rho}^{\alpha}\left(\dot{\psi}^{\alpha T}-\eta^{\alpha T}\frac{\dot{\vartheta}^{\alpha, T}}{(\vartheta^{\alpha T})^2}\right)+\bar{\tau}^{\alpha}_{ji}\tilde{v}^{\alpha}_{i,j}-\overline{v_i^{\alpha''}}\bar{p}^{\alpha}_{,i}-\bar{\rho}^{\alpha}\epsilon^{\alpha}+\bar{\rho}^{\alpha}s^{\alpha}-\frac{1}{\vartheta^{\alpha T}}K^{\alpha}_i\vartheta^{\alpha T}_{,i}\right]$$
$$+\vartheta^{fT}\left[-\bar{\rho}^{f}\left(\dot{\psi}^{fT}-\eta^{fT}\frac{\dot{\vartheta}^{fT}}{(\vartheta^{fT})^2}\right)+t^{fT}_{ji}\tilde{v}^{f}_{i,j}-\overline{v_i^{f''}}\bar{p}^{f}_{,i}-\bar{\rho}^{f}\epsilon^{f}+\bar{\rho}^{f}s^{f}-\frac{1}{\vartheta^{fT}}K^{f}_i\vartheta^{fT}_{,i}+\frac{1}{\vartheta^{fT}}E^{f}_{i,i}\right]\geqslant 0, \quad [57]$$

where Q^{α}_i and Q^{f}_i are the total heat flux vectors defined as

$$Q^{\alpha}_i = \bar{q}^{\alpha}_i + q^{\alpha T}_i, \quad Q^{f}_i = \bar{q}^{f}_i + q^{fT}_i. \quad [58]$$

The constitutive independent variables are

$$\bar{\rho}^{\alpha}, \bar{\rho}^{f}, \mathfrak{J}^{\alpha}, \vartheta^{\alpha T}, \mathfrak{J}^{f}, \vartheta^{fT}, \mathfrak{J}^{\alpha}_{,i}, \vartheta^{\alpha T}_{,i}, \mathfrak{J}^{f}_{,i}, \vartheta^{fT}_{,i}, \bar{d}^{\alpha}_{ij}, \bar{d}^{f}_{ij}, \epsilon^{\alpha}, \epsilon^{f}. \quad [59]$$

These are all frame-indifferent tensors and $\bar{d}^{\alpha}_{ij}$ and $\bar{d}^{f}_{ij}$ are the mean deformation rate tensors defined as

$$\bar{d}^{\alpha}_{ij} = \tfrac{1}{2}(\tilde{v}^{\alpha}_{i,j}+\tilde{v}^{\alpha}_{j,i}), \quad \bar{d}^{f}_{ij} = \tfrac{1}{2}(\tilde{v}^{f}_{i,j}+\tilde{v}^{f}_{j,i}). \quad [60]$$

Along the line of Ahmadi (1989), the following set of frame-indifferent constitutive equations are proposed:

$$\hat{\psi}^{\alpha} = \hat{\psi}^{\alpha}(\bar{\rho}^{\alpha}, \mathfrak{J}^{\alpha}), \qquad \psi^{\alpha T} = \psi^{\alpha T}(\bar{\rho}^{\alpha}, \vartheta^{\alpha T}, \epsilon^{\alpha}),$$
$$\hat{\psi}^{f} = \hat{\psi}^{f}(\bar{\rho}^{f}, \mathfrak{J}^{f}), \qquad \psi^{fT} = \psi^{fT}(\bar{\rho}^{f}, \vartheta^{\alpha T}, \epsilon^{f}),$$
$$\bar{\tau}^{\alpha}_{ij} = \bar{\tau}^{\alpha}_{ij}(\bar{\rho}^{\alpha}, \vartheta^{\alpha}, \bar{d}^{\alpha}_{ij}), \qquad \hat{t}^{\alpha}_{ij} = \hat{t}^{\alpha}_{ij}(\bar{\rho}^{\alpha}, \vartheta^{\alpha T}, \bar{d}^{\alpha}_{ij}, \epsilon^{\alpha}),$$
$$\hat{t}^{f}_{ij} = \bar{t}^{f}_{ij}(\bar{\rho}^{f}, \mathfrak{J}^{f}, \bar{d}^{f}_{ij}), \qquad t^{fT}_{ij} = t^{fT}_{ij}(\bar{\rho}^{f}, \vartheta^{fT}, \bar{d}^{f}_{ij}, \epsilon^{f}),$$

$$
\begin{aligned}
&Q_i^\alpha = Q_i^\alpha(\bar\rho^\alpha, \mathfrak{I}^\alpha, \mathfrak{I}^\alpha_{,i}), \quad K_i^\alpha = K_i^\alpha(\bar\rho^\alpha, \vartheta^{\alpha T}, \vartheta^{\alpha T}_{,i}, \epsilon^\alpha),\\
&Q_i^f = Q_i^f(\bar\rho^f, \mathfrak{I}^f, \mathfrak{I}^f_{,i}), \quad K_i^f = K_i^f(\bar\rho^f, \vartheta^{fT}, \vartheta^{fT}_{,i}, \epsilon^f),\\
&\overline{v_i^{\alpha''}} = \overline{v_i^{\alpha''}}(\bar\rho^\alpha, \bar\rho^\alpha_{,i}, \mathfrak{I}^\alpha, \mathfrak{I}^\alpha_{,i}, \vartheta^{\alpha T}, \epsilon^\alpha),\\
&\overline{v_i^{f''}} = \overline{v_i^{f''}}(\bar\rho^f, \bar\rho^f_{,i}, \mathfrak{I}^f, \mathfrak{I}^f_{,i}, \vartheta^{fT}, \epsilon^f),\\
&E_i^f = E_i^f(\bar\rho^f, \vartheta^{fT}, \vartheta^{fT}_{,i}, \epsilon^f, \epsilon^f_{,i}).
\end{aligned} \quad [61]
$$

Note that for incompressible constituents the respective densities could be replaced by the corresponding volume fraction in the constitutive relations given by [61]. Strictly speaking, according to the principle of equipresence of continuum mechanics all the constitutive dependent variables must, in general, be functions of all the independent constitutive variables. For simplicity of analysis, this principle was not fully utilized and only the most relevant variables are included in the constitutive equations given by [61]. Note also that should additional independent variables be included in the expressions for the free energy functions, the entropy inequality will force the reduction to the forms given in [61]. However, this additional mathematical exercise is not considered here. Employing [61], [57] may be restated as

$$
\begin{aligned}
&\sum_{\alpha=1}^{n} \mathfrak{I}^\alpha\Bigg[-\bar\rho^\alpha\left(\frac{\partial\hat\psi^\alpha}{\partial\mathfrak{I}^\alpha} - \frac{\hat\eta^\alpha}{(\mathfrak{I}^\alpha)^2}\right)\dot{\mathfrak{I}}^\alpha - \frac{1}{\mathfrak{I}^\alpha}Q_i^\alpha\mathfrak{I}^\alpha_{,i} + (\hat t^\alpha_{ji} + \bar p^\alpha\delta_{ij})\tilde v^\alpha_{j,i} + \bar\rho^\alpha\epsilon^\alpha\\
&\quad - \tilde v_i^\alpha \bar P_i^\alpha - \bar\rho^\alpha s^\alpha - \bar\rho^\alpha\overline{v^{\alpha''}_{i,i}} + \bar e^{\alpha+} - \frac{1}{\mathfrak{I}^\alpha}\bar\eta^{\alpha+}\Bigg]\\
&\quad + \mathfrak{I}^f\Bigg[-\bar\rho^f\left(\frac{\partial\hat\psi^f}{\partial\mathfrak{I}^f} - \frac{\hat\eta^f}{(\mathfrak{I}^f)^2}\right)\dot{\mathfrak{I}}^f - \frac{1}{\mathfrak{I}^f}Q_i^f\mathfrak{I}^f_{,i} + (\bar t^f_{ji} + \bar p^f\delta_{ij})\tilde v^f_{j,i} + \bar\rho^f\epsilon^f\\
&\quad - \tilde v_i^f \bar P_i^f - \bar\rho^f s^f - \bar\rho^f\overline{v^{f''}_{i,i}} + \bar e^{f+} - \frac{1}{\mathfrak{I}^f}\bar\eta^{f+}\Bigg]\\
&\quad + \sum_{\alpha=1}^{n}\vartheta^{\alpha T}\Bigg[-\bar\rho^\alpha\left(\frac{\partial\psi^{\alpha T}}{\partial\vartheta^{\alpha T}} - \frac{\eta^{\alpha T}}{(\vartheta^{\alpha T})^2}\right)\dot\vartheta^{\alpha T} + (\tau^\alpha_{ji} + p^{\alpha T}\delta_{ij})\tilde v^\alpha_{i,j}\\
&\quad - \bar\rho^\alpha\epsilon^\alpha - \overline{v_i^{\alpha''}}\bar\rho^\alpha_{,i} + \bar\rho^\alpha s^\alpha - \frac{1}{\vartheta^{\alpha T}}K_i^\alpha\vartheta^{\alpha T}_{,i} - \bar\rho^\alpha\frac{\partial\psi^{\alpha T}}{\partial\epsilon^\alpha}\dot\epsilon^\alpha\Bigg]\\
&\quad + \vartheta^{fT}\Bigg[-\bar\rho^f\left(\frac{\partial\psi^{fT}}{\partial\vartheta^{fT}} - \frac{\eta^{fT}}{(\vartheta^{fT})^2}\right)\dot\vartheta^{fT} + (t^{fT}_{ji} + p^{fT}\delta_{ij})\tilde v^f_{i,j}\\
&\quad - \bar\rho^f\epsilon^f - \overline{v_i^{f''}}\bar\rho^f_{,i} + \bar\rho^f s^f - \frac{1}{\vartheta^{fT}}K_i^f\vartheta^{fT}_{,i} + \frac{1}{\vartheta^{fT}}E^f_{i,i} - \bar\rho^f\frac{\partial\psi^{fT}}{\partial\epsilon^f}\dot\epsilon^f\Bigg] \geqslant 0,
\end{aligned} \quad [62]
$$

where

$$
\begin{aligned}
&\bar p^\alpha = (\bar\rho^\alpha)^2\frac{\partial\hat\psi^\alpha}{\partial\bar\rho^\alpha}, \quad p^{\alpha T} = (\bar\rho^\alpha)^2\frac{\partial\psi^{\alpha T}}{\partial\bar\rho^\alpha},\\
&\bar p^f = (\bar\rho^f)^2\frac{\partial\hat\psi^f}{\partial\bar\rho^f}, \quad p^{fT} = (\bar\rho^f)^2\frac{\partial\psi^{fT}}{\partial\bar\rho^f}.
\end{aligned} \quad [63]
$$

Here, certain consistency conditions for pressures are required. These are

$$
p^{\alpha T} = \gamma^\alpha\bar\rho^\alpha k^\alpha, \quad p^{fT} = \tfrac{2}{3}\bar\rho^f k^f, \quad [64]
$$

where γ^α is a function of the solid volume fraction v^α.

For a dispersed mixture when the particles are not in direct contact except during the relatively short period of collision and surface tension and Brownian motion effects are negligible, it may be assumed that

$$
\bar p^\alpha = v^\alpha p^f, \quad \bar p^f = v^f p^f, \quad [65]
$$

where p^f is the mean pressure in the fluid phase. (For a fully-saturated solid–liquid mixture, this assumption will be justified in a latter section.) According to McTique *et al.* (1986) and Ahmadi

(1987), there are reasons to suggest that the particulate phase mean pressure exceeds that of the surrounding fluid in direct proportion to the square of the mean velocity differences. Ahmadi (1987) also has shown that with an appropriate choice of $\hat{\psi}^\alpha$ one could derive the pressure differences as well as the virtual mass forces. However, this further refinement is not considered here and [65] will be used.

Equations [63] and [64] impose additional restrictions on the expressions for the mean free energy functions. Demanding that the entropy inequality [62] holds for all independent variations of $\bar{\vartheta}^\alpha$, $\bar{\vartheta}^f$, $\vartheta^{\alpha T}$ and ϑ^{fT}, it follows that

$$\hat{\eta}^\alpha = (\bar{\vartheta}^\alpha)^2 \frac{\partial \hat{\psi}^\alpha}{\partial \bar{\vartheta}^\alpha}, \quad \eta^{\alpha T} = (\vartheta^{\alpha T})^2 \frac{\partial \psi^{\alpha T}}{\partial \vartheta^{\alpha T}},$$

$$\hat{\eta}^f = (\bar{\vartheta}^f)^2 \frac{\partial \hat{\psi}^f}{\partial \bar{\vartheta}^f}, \quad \eta^{fT} = (\vartheta^{fT})^2 \frac{\partial \psi^{fT}}{\partial \vartheta^{fT}}. \qquad [66]$$

The mean temperatures are defined as

$$\hat{\theta}^\alpha = \frac{1}{\bar{\vartheta}^\alpha}, \quad \hat{\theta}^f = \frac{1}{\bar{\vartheta}^f}. \qquad [67]$$

Note that $\hat{\theta}^\alpha$ and $\hat{\theta}^f$ are, in general, different from the mean temperatures $\bar{\theta}^\alpha$ and $\bar{\theta}^f$. They are equal only within the limit of a linearized theory.

According to Ahmadi (1989) the coldness of turbulence must be inversely proportional to its kinetic energy. Thus,

$$\vartheta^{\alpha T} = \frac{C^{\alpha T}}{k^\alpha}, \quad \vartheta^{fT} = \frac{C^{fT}}{k^f}, \qquad [68]$$

where $C^{\alpha T}$ and C^{fT} are some positive parameters corresponding to the energy capacities of turbulent fluctuations with $C^{fT} = C_0^{fT}\epsilon^f$ and $C_0^{fT} \geqslant 0$. Employing [66]–[68], [62] reduces to

$$\begin{aligned}
&\sum_{\alpha=1}^{n} \frac{1}{\bar{\theta}^\alpha}\left[\frac{1}{\bar{\theta}^\alpha} Q_i^\alpha \hat{\theta}^\alpha_{,i} + (\hat{t}^\alpha_{ji} + \bar{p}^\alpha \delta_{ij})\tilde{v}^\alpha_{j,i} + \bar{\rho}^\alpha \epsilon^\alpha - \tilde{v}^\alpha_i \bar{P}^\alpha_i - \bar{\rho}^\alpha s^\alpha - \bar{p}^\alpha \overline{v^{\alpha''}_{i,i}} + \bar{e}^{\alpha +} - \hat{\theta}^\alpha \bar{\eta}^{\alpha +}\right] \\
&+ \frac{1}{\bar{\theta}^f}\left[\frac{1}{\bar{\theta}^f} Q_i^f \hat{\theta}^f_{,i} + (\bar{t}^f_{ji} + \bar{p}^f \delta_{ij})\tilde{v}^f_{j,i} + \bar{\rho}^f \epsilon^f - \tilde{v}^f_i \bar{P}^f_i - \bar{\rho}^f s^f - \bar{p}^f \overline{v^{f''}_{i,i}} + \bar{e}^{f+} - \hat{\theta}^f \bar{\eta}^{f+}\right] \\
&+ \sum_{\alpha=1}^{n} \frac{C^{\alpha T}}{k^\alpha}\left[(\bar{\tau}^\alpha_{ji} + \gamma^\alpha \bar{\rho}^\alpha k^\alpha \delta_{ij})\tilde{v}^\alpha_{i,j} - \bar{\rho}^\alpha \epsilon^\alpha - \overline{v^{\alpha''}_i}\bar{p}^\alpha_{,i} + \bar{\rho}^\alpha s^\alpha + \frac{1}{e^\alpha} K_i^\alpha k^\alpha_{,i} - \bar{\rho}^\alpha \frac{\partial \psi^{\alpha T}}{\partial \epsilon^\alpha}\dot{\epsilon}^\alpha\right] \\
&+ \frac{C^{fT}}{k^f}\left[(t^{fT}_{ji} + \tfrac{2}{3}\bar{\rho}^f k^f \delta_{ij})\tilde{v}^f_{i,j} - \bar{\rho}^f \epsilon^f - \overline{v^{f''}_i}\bar{p}^f_{,i} + \bar{\rho}^f s^f + \frac{1}{k^f} K_i^f\left(k^f_{,i} - \frac{k^f}{\epsilon^f}\epsilon^f_{,i}\right)\right. \\
&\left. + \frac{k^f}{C^{fT}} E^f_{i,i} - \bar{\rho}^f \frac{\partial \psi^{fT}}{\partial \epsilon^f}\dot{\epsilon}^f\right] \geqslant 0. \qquad [69]
\end{aligned}$$

Inequality [69] imposes important thermodynamical restrictions on the admissible forms of constitutive equations. Many dependent and independent constitutive variables appear as simple products in the entropy inequality given by [69]. Therefore, linear constitutive equations consistent with the Clausius–Duhem inequality may be formulated. In the following, formulations of isotropic quasi-linear constitutive equations are considered.

Assuming that the stresses are linear functions of the corresponding mean deformation rate fields, it follows that

$$\hat{t}^\alpha_{ij} = -(\bar{p}^\alpha + \tfrac{2}{3}\mu^\alpha \tilde{d}^\alpha_{mm})\delta_{ij} + 2\mu^\alpha \tilde{d}^\alpha_{ij}, \qquad [70]$$

$$\bar{\tau}^\alpha_{ij} = -(\gamma^\alpha \bar{\rho}^\alpha k^\alpha + \tfrac{2}{3}\mu^{\alpha T}\tilde{d}^\alpha_{mm})\delta_{ij} + 2\mu^{\alpha T}\tilde{d}^\alpha_{ij}, \qquad [71]$$

$$\bar{t}^f_{ij} = -(\bar{p}^f + \tfrac{2}{3}\mu^f \tilde{d}^f_{mm})\delta_{ij} + 2\mu^f \tilde{d}^f_{ij} \qquad [72]$$

and

$$t^{fT}_{ij} = -\tfrac{2}{3}(\bar{\rho}^f k^f + \mu^{fT}\tilde{d}^f_{mm})\delta_{ij} + 2\mu^{fT}\tilde{d}^f_{ij}. \qquad [73]$$

Equations [70]–[73] are extensions of the Boussinesq constitutive equation for the Reynolds stress tensor to multicomponent systems. In these equations μ^α and μ^f are the coefficients of viscosity which are functions of the solid volume fractions, and $\mu^{\alpha T}$ and μ^{fT} are the coefficients of turbulence (eddy) viscosity. Along the lines of the Kolmogorov–Prandtl hypothesis (Launder & Spalding 1972; Rodi 1982), it is assumed that

$$\mu^{\alpha T} = C^{\alpha\mu}\bar{\rho}^\alpha l^\alpha (k^\alpha)^{1/2}, \quad \mu^{fT} = \frac{C^{f\mu}\bar{\rho}^f (k^f)^2}{\epsilon^f}, \qquad [74]$$

where $C^{\alpha\mu}$ and $C^{f\mu}$ are parameters which depend on v^α and v^f, and l^α is an appropriate length scale of the αth particulate phase. The choice of length scale for a particulate phase needs further elaboration. For relatively dense collision-dominated mixture flows, the diameter d^α is the natural length scale. Such a choice may be justified based on the kinetic models of Lun *et al.* (1984) and Ahmadi & Ma (1986). For dilute mixtures, the fluid turbulence dominates and the particles are essentially transported by the fluid phase. Hence, the fluid length scale is the only relevant scale (Besnard & Harlow 1985). For intermediate ranges of solid volume fractions, turbulence is mainly generated by particle–fluid interactions. In this case the most relevant scale is the mean interparticle distance. Note also that the Stokes assumption is also used for the second coefficients of viscosity in constitutive equations [70]–[73].

The constitutive equations for the density–velocity correlations are given by

$$\overline{v_i^{\alpha''}} = -\frac{\mu^{\alpha T}}{\sigma^{\alpha p}\bar{\rho}^\alpha k^\alpha}\bar{p}^\alpha_{,i} \qquad [75]$$

and

$$\overline{v_i^{f''}} = -\frac{\mu^{fT}}{\sigma^{fp}\bar{\rho}^f k^f}\bar{p}^f_{,i}, \qquad [76]$$

where $\sigma^{\alpha p}$ and σ^{fp} are parameters which are, in general, functions of v^α and v^f. Equations [75] and [76] are generalizations of that proposed by Ahmadi (1989) for single-phase compressible turbulent flows.

The fluctuation energy fluxes are assumed to be given as

$$K_i^\alpha = \frac{\mu^{\alpha T}}{\sigma^{\alpha k}}k^\alpha_{,i}, \quad K_i^f = \left(\mu^f + \frac{\mu^{fT}}{\sigma^{fk}}\right)\left(k^f_{,i} - \frac{k^f}{\epsilon^f}\epsilon^f_{,i}\right), \qquad [77]$$

where $\sigma^{\alpha k}$ and σ^{fk} are parameters corresponding to the Prandtl numbers for turbulence energy fluxes. The expression for the fluid energy flux, given by [77], contains an additional term which depends on the gradient of the dissipation rate. Such a form was obtained earlier by Yoshizawa (1985) from a rigorous analysis for a single-phase fluid turbulence.

We assume that the heat fluxes satisfy the extended Fourier law of conduction, i.e.

$$Q_i^\alpha = (\kappa^\alpha + \kappa^{\alpha T})\,\hat{\theta}^\alpha_{,i}, \quad Q_i^f = (\kappa^f + \kappa^{fT})\,\hat{\theta}^f_{,i}, \qquad [78]$$

where κ is the heat conductivity and the superscript T refers to turbulence.

The constitutive equations given by [70]–[78] are compatible with the averaged Clausius–Duhem inequality given by [69]. The entropy inequality also imposes the following restriction on the parameters:

$$\begin{aligned} &\mu^\alpha \geqslant 0, \quad \mu^f \geqslant 0, \quad \mu^{\alpha T} \geqslant 0, \quad \mu^{fT} \geqslant 0, \\ &\kappa^\alpha \geqslant 0, \quad \kappa^f \geqslant 0, \quad \kappa^{\alpha f} \geqslant 0, \quad \kappa^{fT} \geqslant 0, \\ &\sigma^{\alpha p} \geqslant 0, \quad \sigma^{fp} \geqslant 0, \quad \sigma^{\alpha k} \geqslant 0, \quad \sigma^{fk} \geqslant 0. \end{aligned} \qquad [79]$$

To derive the required constitutive equations for the remaining terms, we restrict ourselves to isothermal mixtures. The interaction momentum supply terms must satisfy the following entropy inequality for an isothermal mixture:

$$-\tilde{v}_i^f \bar{P}_i^f - \sum_{\alpha=1}^{n} \tilde{v}_i^\alpha \bar{P}_i^\alpha \geqslant 0. \qquad [80]$$

Using [32], [80] may be restated as

$$\sum_{\alpha=1}^{n} \bar{P}_i^{\alpha}(\tilde{v}_i^{\mathrm{f}} - \tilde{v}_i^{\alpha}) \geqslant 0. \qquad [81]$$

The mean interaction momentum supply for the αth particulate phase satisfying [81] is given by

$$\bar{P}_i^{\alpha} = D_{ij}^{\alpha}(\tilde{v}_j^{\mathrm{f}} - \tilde{v}_j^{\alpha}), \qquad [82]$$

where D_{ij}^{α} is a positive definite matrix given as

$$D_{ij}^{\alpha} = D_0^{\alpha}\delta_{ij} + 2L^{\alpha}\bar{d}_{ij}^{\mathrm{f}}. \qquad [83]$$

Here, D_0^{α} and L^{α} correspond to the drag and shear lift coefficients. For a dilute suspension of spherical particles in an incompressible fluid of density ρ^{f} and viscosity μ_0^{f},

$$D_0^{\alpha} = \frac{18\mu_0^{\mathrm{f}}\nu^{\alpha}}{(d^{\alpha})^2}, \quad L^{\alpha} = \frac{2.594(\rho^{\mathrm{f}}\mu_0^{\mathrm{f}})^{1/2}\nu^{\alpha}}{d^{\alpha}(\bar{d}_{ij}^{\mathrm{f}}\bar{d}_{ji}^{\mathrm{f}})^{1/4}} \qquad [84]$$

were suggested in the work of Drew (1976), Ahmadi (1982) and McTique *et al.* (1986). Equation [84] is consistent with the lift force for a single sphere in a uniform shear field, as obtained by Saffman (1965). Here, d^{α} is the diameter of the αth particulate phase. Equation [84] also assumes that the particles are in Stokes flow regime. When the particle Reynolds number,

$$\mathrm{Re}_{\mathrm{d}}^{\alpha} = \frac{\rho^{\mathrm{f}} d^{\alpha}|\tilde{\mathbf{v}}^{\mathrm{f}} - \tilde{\mathbf{v}}^{\alpha}|}{\mu_0^{\mathrm{f}}}, \qquad [85]$$

is not sufficiently small, the drag coefficient becomes

$$D_0^{\alpha} = \frac{18\mu_0^{\mathrm{f}}\nu^{\alpha}}{(d^{\alpha})^2}\frac{[1 + 0.1(\mathrm{Re}_{\mathrm{d}}^{\alpha})^{0.75}]}{\left(1 - \dfrac{\nu^{\alpha}}{\nu_{\mathrm{m}}^{\alpha}}\right)^{2.5\nu_{\mathrm{m}}^{\alpha}}}, \qquad [86]$$

where the modifications for nondilute flows are also included. Here, $\nu_{\mathrm{m}}^{\alpha}$ is the limiting dense packing volume fraction for shear flows. For a single size spherical particulate phase, $\nu_{\mathrm{m}} = 0.64356$ (Ma & Ahmadi 1986). The expression for the lift coefficient for dense or high Re_{d} flows is not available as yet.

The constitutive equation for the mean interaction momentum supply of the fluid phase may be obtained from [32] and [82], i.e.

$$\bar{P}_i^{\mathrm{f}} = \sum_{\alpha=1}^{n} D_{ij}^{\alpha}(\tilde{v}_j^{\alpha} - \tilde{v}_j^{\mathrm{f}}). \qquad [87]$$

Constitutive equations given by [82] and [87] assume that the momentum only transfers between the fluid and the particulate phases. The interaction momentum supply due to the collisions of particles of different species is, thus, neglected. Whenever such interactions become important (e.g. very dense multiphase flows or mixtures of granular particles), [82] and [87] must be modified accordingly (e.g. by replacing the $\tilde{v}_i^{\mathrm{f}} - \tilde{v}_i^{\alpha}$ term in [82] with $\tilde{v}_i - \tilde{v}_i^{\alpha}$).

Entropy inequality [69] imposes the following restriction on the fluctuation energy source terms:

$$\bar{\rho}^{\mathrm{f}} s^{\mathrm{f}}\left(\frac{C^{\mathrm{fT}}}{k^{\mathrm{f}}} - \frac{1}{\bar{\theta}^{\mathrm{f}}}\right) + \sum_{\alpha=1}^{n} \bar{\rho}^{\alpha} s^{\alpha}\left(\frac{C^{\alpha\mathrm{T}}}{k^{\alpha}} - \frac{1}{\bar{\theta}^{\alpha}}\right) \geqslant 0. \qquad [88]$$

We assume that the fluctuation energy interactions are mainly between the fluid and particulate phases. Furthermore,

$$\bar{\rho}^{\mathrm{f}} s^{\mathrm{f}} = -\sum_{\alpha=1}^{n} \bar{\rho}^{\alpha} s^{\alpha}, \qquad [89]$$

and the direct fluctuation energy transfer between particulate phases is negligible. Note that [89] assumes that the fluctuation energies are simply exchanged between the phases and ignores the associated dissipations. However, this need not be of much concern since these additional dissipations could be added to their respective ϵ terms and be modeled as part of the total dissipation rates. For an isothermal mixture, using [89], [88] may be restated as

$$\sum_{\alpha=1}^{n} \frac{\bar{\rho}^{\alpha} s^{\alpha}}{k^{\alpha} k^{\mathrm{f}}} (C^{\alpha\mathrm{T}} k^{\mathrm{f}} - C^{\mathrm{fT}} k^{\alpha}) \geqslant 0. \qquad [90]$$

The constitutive equations for the fluctuation energy supply for the αth particulate phase, which is consistent with entropy equality [90], is given as

$$\bar{\rho}^{\alpha} s^{\alpha} = s_0^{\alpha} (C^{\alpha\mathrm{T}} k^{\mathrm{f}} - C^{\mathrm{fT}} k^{\alpha}), \qquad [91]$$

with

$$s_0^{\alpha} \geqslant 0 \qquad [92]$$

being a material parameter. The expression for the fluid turbulence energy supply then becomes

$$\bar{\rho}^{\mathrm{f}} s^{\mathrm{f}} = \sum_{\alpha=1}^{n} s_0^{\alpha} (C^{\mathrm{fT}} k^{\alpha} - C^{\alpha\mathrm{T}} k^{\mathrm{f}}). \qquad [93]$$

The energy supply terms given by [91] and [93] are quite similar to the expressions obtained by Kashiwa (1987) via a statistical analysis. The only difference is the presence of additional source terms which are proportional to the mean velocity differences in the formulation of Kashiwa (1987) and Besnard & Harlow (1985). These terms, which are expected to become significant for nearly laminar flows, are neglected in the present formulation.

The constitutive relationships for multiphase turbulent flows given by [70]–[78], [81], [87], [91] and [93] are consistent with the averaged Clausius–Duhem inequality.

DISSIPATION

Thermodynamically consistent algebraic expressions or transport equations for the dissipation rates were suggested by Ma (1987) and Ahmadi (1989). Here only the simple algebraic expressions are considered. Accordingly:

$$\epsilon^{\alpha} = a^{\alpha} (k^{\alpha})^{3/2}, \qquad [94]$$

where the parameter a^{α} has a dimension of inverse length. Equation [94] is in agreement with the expression for the collisional dissipation rate obtained from the kinetic theories of granular materials. Note also that ϵ^{α} now includes the dissipative parts of the fluctuation energy interaction term. Similarly,

$$\epsilon^{\mathrm{f}} = a^{\mathrm{f}} (k^{\mathrm{f}})^{3/2}. \qquad [95]$$

This simple algebraic expression is compatible with the expression used in the so-called one-equation turbulence model with

$$a^{\mathrm{f}} = \frac{C^{\mathrm{fD}}}{\Lambda^{\mathrm{f}}}, \qquad [96]$$

where C^{fD} is a material parameter (constant for dilute mixtures) and Λ^{f} is a length macroscale of turbulence.

INCOMPRESSIBLE MIXTURES

For fully-saturated isothermal mixtures, when the fluid and particulate constituents are incompressible, additional constraints given by [19], [21] and [22] are imposed on the motion. The free energy functions $\hat{\psi}^{\alpha}$ and $\hat{\psi}^{\mathrm{f}}$ given in [61] are no longer functions of the respective densities and the thermodynamical pressures $\bar{p}^{\alpha}$ and $\bar{p}^{\mathrm{f}}$, given in [63], remain undefined. In this section, the required modifications are briefly described.

Adding [19] and [21] for all the species and using [22], we find

$$(v^{\mathrm{f}} \tilde{v}_k^{\mathrm{f}})_{,k} + \sum_{\alpha=1}^{n} (v^{\alpha} \tilde{v}_k^{\alpha})_{,k} = 0. \qquad [97]$$

The constraint given by [97] must be incorporated in the entropy inequality given by [57] or [62]. Multiplying [97] by a Lagrangian multiplier p^{f} (letter to be identified as the mean pressure in the fluid phase) and adding the result to [57], [62] remains essentially valid with the following minor modifications: the pressure terms are now given by [65] and $\bar{P}_i^{\alpha}$ must be replaced by $\bar{P}_i^{\alpha} - p^{f}v^{\alpha}_{,i}$, with a similar change for the fluid phase, i.e. the constraint given by [80] must be replaced by

$$-\tilde{v}_i^{f}(\bar{P}_i^{f} - p^{f}v^{f}_{,i}) - \sum_{\alpha=1}^{n} \tilde{v}_i^{\alpha}(\bar{P}_i^{\alpha} - p^{f}v^{\alpha}_{,i}) \geqslant 0. \qquad [98]$$

As a result, the constitutive equations given by [82] and [87] now become

$$\bar{P}_i^{\alpha} = D_{ij}^{\alpha}(\tilde{v}_j^{f} - \tilde{v}_j^{\alpha}) + p^{f}v^{\alpha}_{,i} \qquad [99]$$

and

$$\bar{P}_i^{f} = \sum_{\alpha=1}^{n} D_{ij}^{\alpha}(\tilde{v}_j^{\alpha} - \tilde{v}_j^{f}) + p^{f}v^{f}_{,i}. \qquad [100]$$

Explicit expressions for the equations of motion now become

$$\rho^{\alpha}v^{\alpha}\frac{\mathrm{d}\,\tilde{v}_i^{\alpha}}{\mathrm{d}t} = \rho^{\alpha}v^{\alpha}f_i^{\alpha} - v^{\alpha}\frac{\partial p^{f}}{\partial x_i} - \frac{\partial}{\partial x_i}[\gamma^{\alpha}\rho^{\alpha}v^{\alpha}k^{\alpha} + \tfrac{2}{3}(\mu^{\alpha} + \mu^{\alpha\mathrm{T}})\tilde{v}^{\alpha}_{\mathrm{m,m}}] + \frac{\partial}{\partial x_j}\left[(\mu^{\alpha} + \mu^{\alpha\mathrm{T}})\left(\frac{\partial \tilde{v}_i^{\alpha}}{\partial x_j} + \frac{\partial \tilde{v}_j^{\alpha}}{\partial x_i}\right)\right] + D_{ij}^{\alpha}(\tilde{v}_j^{f} - \tilde{v}_j^{\alpha}) \qquad [101]$$

and

$$\rho^{f}v^{f}\frac{\mathrm{d}\tilde{v}_i^{f}}{\mathrm{d}t} = \rho^{f}v^{f}f_i^{f} - v^{f}\frac{\partial p^{f}}{\partial x_i} - \frac{2}{3}\frac{\partial}{\partial x_i}[\rho^{f}v^{f}k^{f} + (\mu^{f} + \mu^{f\mathrm{T}})\tilde{v}^{f}_{\mathrm{m,m}}] + \frac{\partial}{\partial x_j}\left[(\mu^{f} + \mu^{f\mathrm{T}})\left(\frac{\partial \tilde{v}_i^{f}}{\partial x_j} + \frac{\partial \tilde{v}_j^{f}}{\partial x_i}\right)\right] + \sum_{\alpha=1}^{n} D_{ij}^{\alpha}(\tilde{v}_j^{\alpha} - \tilde{v}_j^{f}), \qquad [102]$$

where [18] and [20] are used. When the fluid and particles are incompressible and the mixture is saturated and isothermal, [35], [40], [101] and [102], together with [19], [21] and [22] form $5(n+1)+1$ equations for determining $5(n+1)+1$ unknowns, $\tilde{v}$s, vs, ks and p^{f} for a fully-saturated turbulent multiphase flow with incompressible constituents.

RAPID GRANULAR FLOWS

For rapid flows of relatively dense heavy particles, the effect of the fluid phase is negligible or secondary and the process of momentum transport is dominated by the particle–particle collisions. Thus, the main stress tensor is given by [71] with the coefficients of viscosity given by [74]. This expression is in agreement with the one obtained from a kinetic formulation by Ahmadi & Ma (1986) and Ma & Ahmadi (1988) for a collection of nearly elastic spherical particles. A detailed comparison for spherical particles shows that

$$C^{\alpha\mu} = 0.0853[(\chi v^{\alpha})^{-1} + 3.2 + 12.1824v^{\alpha}\chi] \qquad [103]$$

and

$$\gamma^{\alpha} = \tfrac{2}{3}(1 + 4v^{\alpha}\chi) + \tfrac{1}{3}(1 - r^{2}), \qquad [104]$$

where r is the coefficient of restitution and the radial distribution χ is given by

$$\chi = \frac{1 + 2.5v^{\alpha} + 4.5904(v^{\alpha})^{2} + 4.515439(v^{\alpha})^{3}}{\left[1 - \left(\frac{v^{\alpha}}{v_{\mathrm{m}}}\right)^{3}\right]^{0.678021}}, \qquad [105]$$

with $v_{\mathrm{m}} = 0.64356$.

Comparing the expression for the fluctuation energy fluxes given by [77] with the corresponding expression developed in the work of Ma & Ahmadi (1988), we find

$$\sigma^{\alpha k} = \frac{1.2[(\chi)^{-1} + 3.2v^{\alpha} + 12.1824(v^{\alpha})^2\chi]}{(1+r^2)[\chi^{-1} + 4.8v^{\alpha} + 12.1184(v^{\alpha})^2\chi]}. \qquad [106]$$

The expression for the dissipation given by [94] is also in agreement with that used by Ahmadi & Ma (1986), with

$$a^{\alpha} = \frac{3.9v^{\alpha}\chi(1-r^2)}{d^{\alpha}} \qquad [107]$$

where d^{α} is the diameter of spherical particles.

The particulate fluctuation energy supply term may also be matched with the one obtained from the kinetic theory, i.e.

$$s_0^{\alpha}(C^{\alpha T}k^{f} - C^{fT}k^{\alpha}) = 2D_0^{\alpha}(ck^{f} - k^{\alpha}), \qquad [108]$$

where the expression on the r.h.s. of [108] was obtained by Ma & Ahmadi (1988) with

$$c = \frac{1}{1 + \dfrac{\bar{\rho}^{\alpha}}{D_0^{\alpha}T_{L}}}; \qquad [109]$$

T_{L} being the Lagrangian time macroscale of fluid turbulence. It appears that [108] holds if

$$s_0^{\alpha}C^{\alpha T} = 2cD_0^{\alpha}, \quad s_0^{\alpha}C^{fT} = 2D_0^{\alpha}. \qquad [110]$$

The results presented in this section show that the present model is in complete agreement with the kinetic model developed by Ma & Ahmadi (1988) for a dense collection of spherical particles, including interstitial fluid effects. Similarly, it may be also shown that the developed turbulence model is also compatible with the kinetic model of Lun *et al.* (1984) for granular particles. The parameters given by [103]–[107] have to be somewhat modified for this purpose.

TWO-PHASE FLOWS

In this section, the special case of a two-phase solid–liquid mixture flow is considered. It is assumed that the mixture is fully-saturated and the constituents are incompressible. With $n = 1$, [19], [21], [22], [35], [40], [65], [101] and [102] are the appropriate governing equations of motion. When the effects of the fluctuation kinetic energy of the particulate phase are neglected and for dilute two-phase flows, these equations become quite similar to those developed by Chen & Wood (1985) and Elghobashi & Abou-Arab (1983). The main difference appears in the fluctuation energy interaction term. In the present model this term is given by [108] (with c given by [109]), which is similar to the expression suggested by Pourahmadi & Humphrey (1982), rather than the exponential form used in the work of Elghobashi & Abou-Arab (1983).

Based on Rotta's proposal, the time macroscale was estimated by Elghobashi & Abou-Arab (1983). In our notation their result may be restated as

$$T_{L} = \frac{0.165k^{f}}{\epsilon^{f}}. \qquad [111]$$

Note that [111] is for the limit of dilute two-phase flows. For dense flows, T_{L} may depend on the solid volume fraction v.

CONCLUSIONS

The technique of thermodynamical formulation in turbulence modeling is extended to the analysis of turbulent flows of dispersed multiphase solid–fluid mixtures. The ensemble-averaging technique is applied directly to the global equations of balance for each constituent and the conservation laws for the mean motions of various phases are developed. In particular, the averaged form of the Clausius–Duhem inequality and the evolution equations for the fluctuation energies of the fluid phase and the particulate constituents are derived. These equations complemented the existing balance laws for the mean turbulent flow field of solid–fluid mixtures. Based on the averaged entropy inequality, constitutive equations for the stresses, energy and heat fluxes of various species are developed and the closed system of equations governing the turbulent flows of multiphase mixtures is obtained. The special case of incompressible constituents is also discussed. It is shown that the model is consistent with the recently developed turbulence models for dilute two-phase flows and dense rapid granular flows in special limiting cases.

The presented multiphase turbulent flow model includes the transport equations for the fluctuation kinetic energies of the particulate constituents in addition to the fluid phase. This allows a more accurate formulation of the expressions for the particulate normal and shear stresses; furthermore, the model becomes applicable to highly dense mixtures for which the particle–particle collisions become significant. It is also shown that the model is consistent with those obtained from the kinetic theories for dense granular flows. Therefore, the model, in principle, could handle the entire range of dilute to dense turbulent mixture flows. In Part II, extensive comparisons of the predictions of this turbulence model with the experimental data for the case of a simple shear flow of a dense mixture are presented.

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